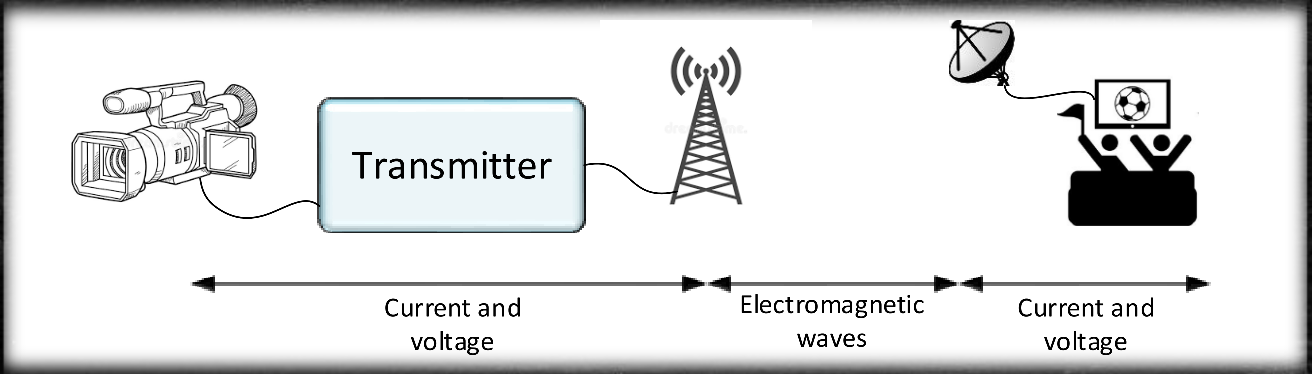
Basic mathematics

Applied in DTV systems

СПбГУТ)))

Definitions

- The information can be expressed as a function of time, $x(t)$
- A periodic function is defined mathematically as
$$x(t) = x(t + T_0)\ \forall t \in \Re$$
  - Periodic functions as sine and cosine will be the basic functions for communication systems
- If a function of time carries information it is called signal
- A signal $x(t)$ can be transmitted as voltage, current, etc.



- The energy of a signal is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

- There are signals with $E = \infty$, as periodic signals for them it is defined the average power:

$$P = \lim_{T_0 \to \infty} \frac{1}{2T_0} \int_{-T_0}^{T_0} |x(t)|^2 \, dt$$

- Signals can be classified in
  - Energy signals: $0 < E < \infty$
  - Power signals: $0 < P < \infty$

- For discrete systems the energy can be expressed as

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

- A periodic signal will have a period of N_0 samples

$$x[n] = x[n + N_0] \; \forall \; n \in \mathbb{Z}$$

- Its power can be expressed as

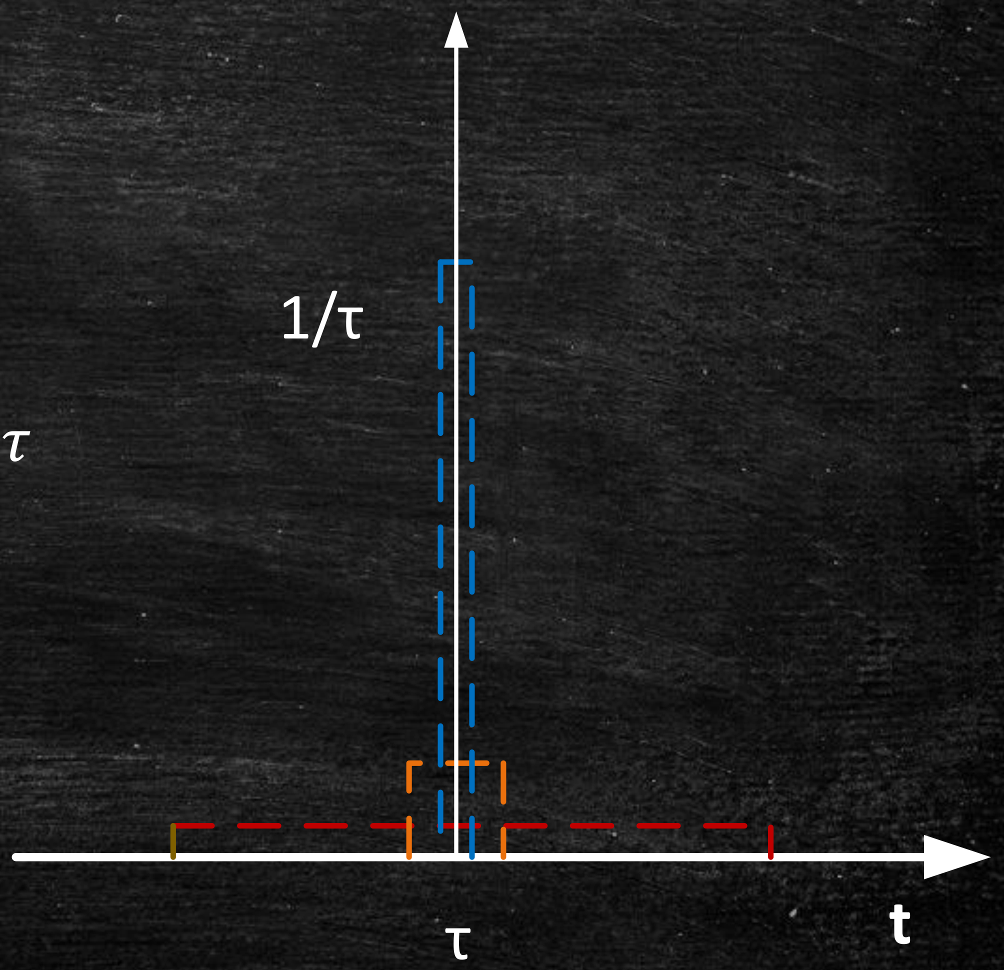
$$E = \lim_{N_0 \to \infty} \frac{1}{2N_0 + 1} \sum_{n=-N_0}^{N_0} |x[n]|^2$$

- A very important function in telecommunications is Dirac delta
  - Mathematically defined as

$$\delta(t)\begin{cases}+\infty, & t=0\\ 0, & t\neq 0\end{cases}$$

  - Like a rectangle of height $1/\tau$ and width τ when $\tau \to 0$
  - Its area is unity

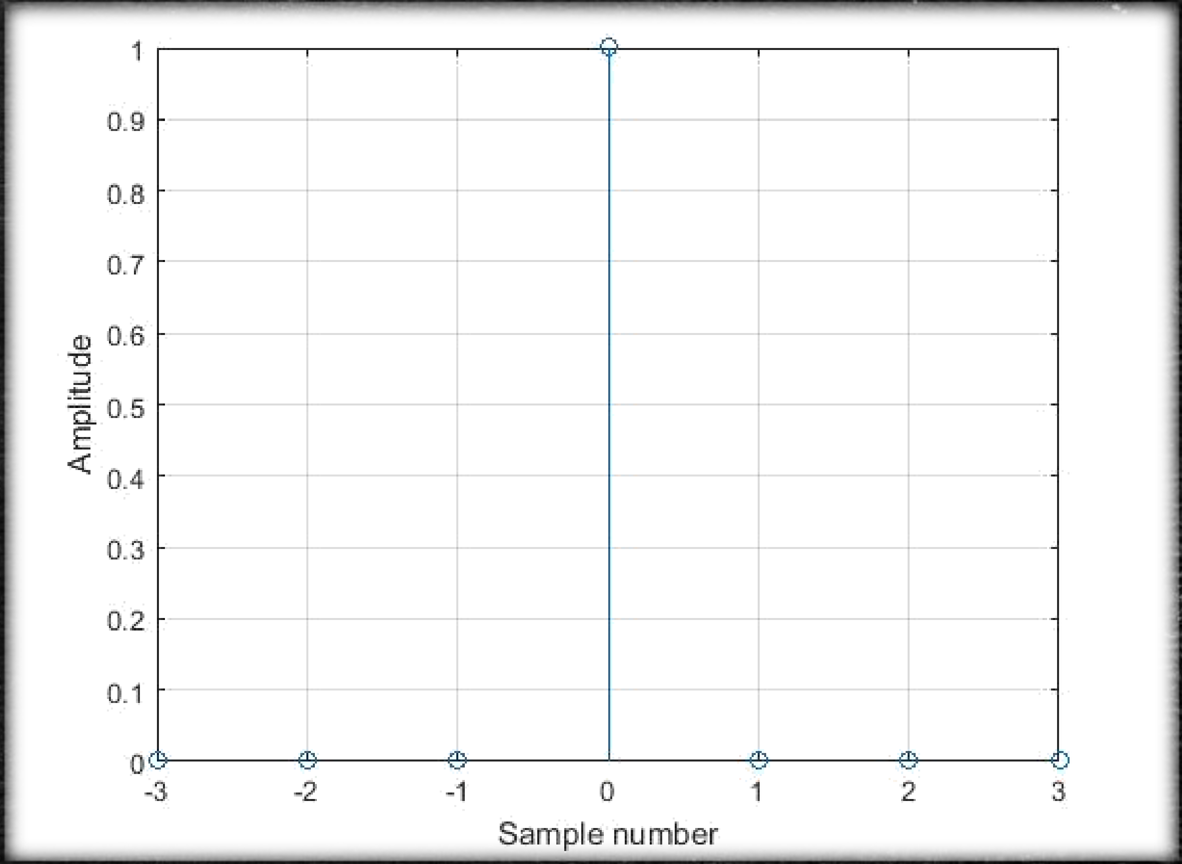
$$\int_{-\infty}^{+\infty}\delta(t)dt = 1$$



- The discrete version of the Dirac delta is much simpler

$$\delta[n]\begin{cases}1, & n=0\\0, & n\neq 0\end{cases}$$

- It has the equivalent characteristics of the continuous one

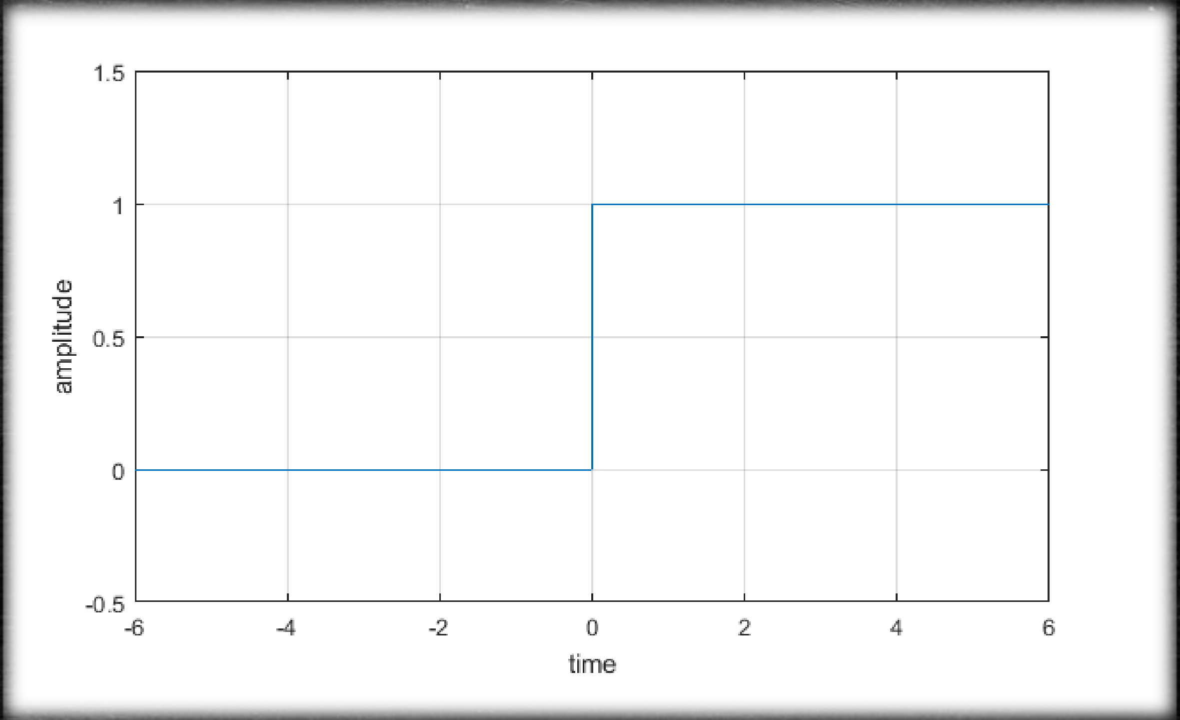


- Heaviside step function
  - Is defined as the primitive of the Dirac delta

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = \begin{cases} 1, t \geq 0 \\ 0, t < 0 \end{cases}$$

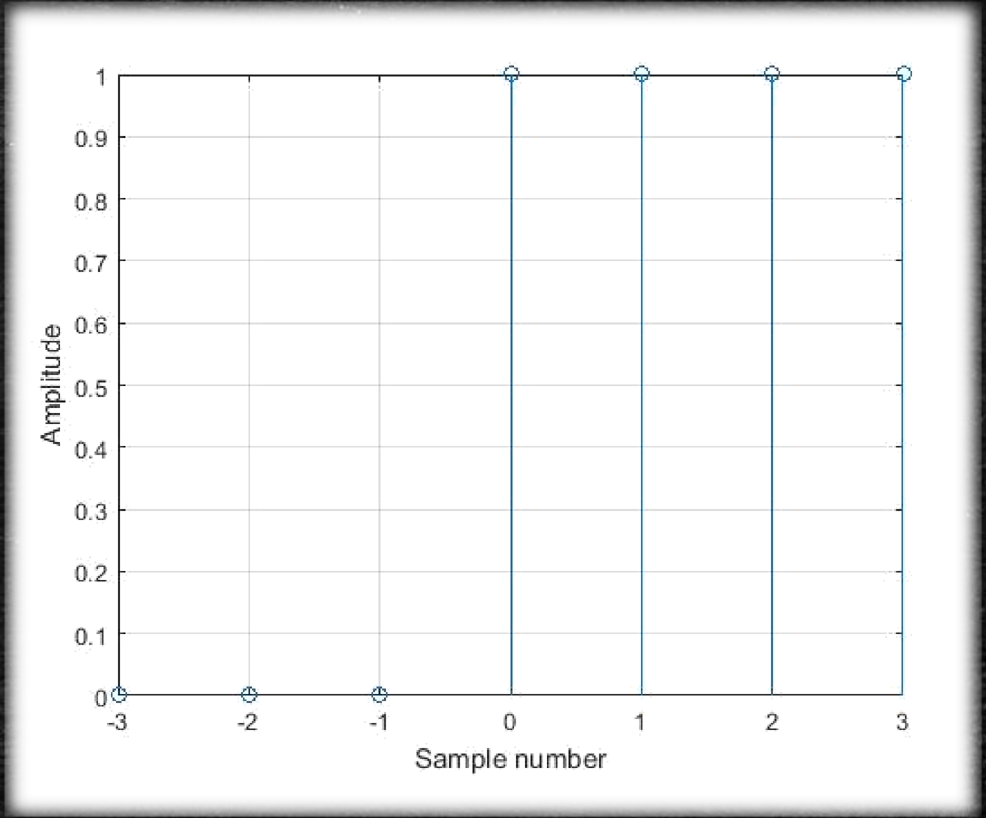
- The Dirac delta allows to define the derivative of noncontinuous functions:

$$\delta(t) = \frac{du(t)}{dt}$$



- The discrete version of the Heaviside step is defined as

$$u[n] = \begin{cases} 1, n \geq 0 \\ 0, n < 0 \end{cases}$$



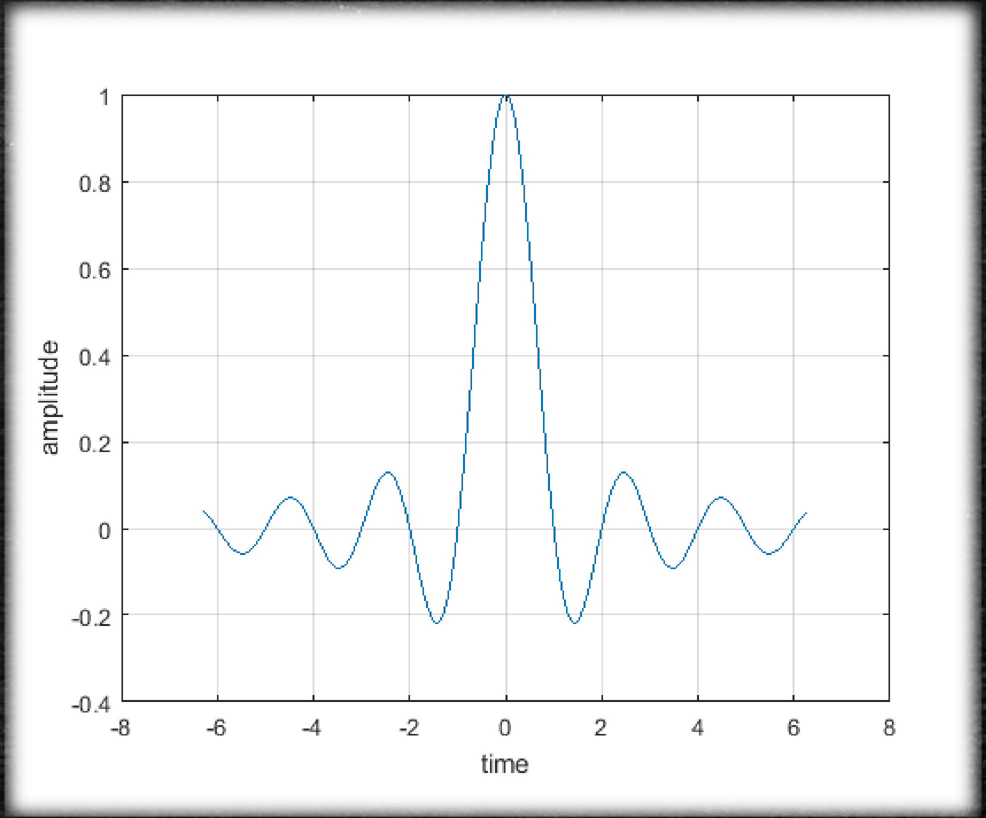
- The sinc function
  - Mathematically defined as

$$sinc(t) = \frac{\sin(\pi t)}{\pi t}$$

  - This function is one of the most used in communications
  - The function takes value of 1 for $t = 0$

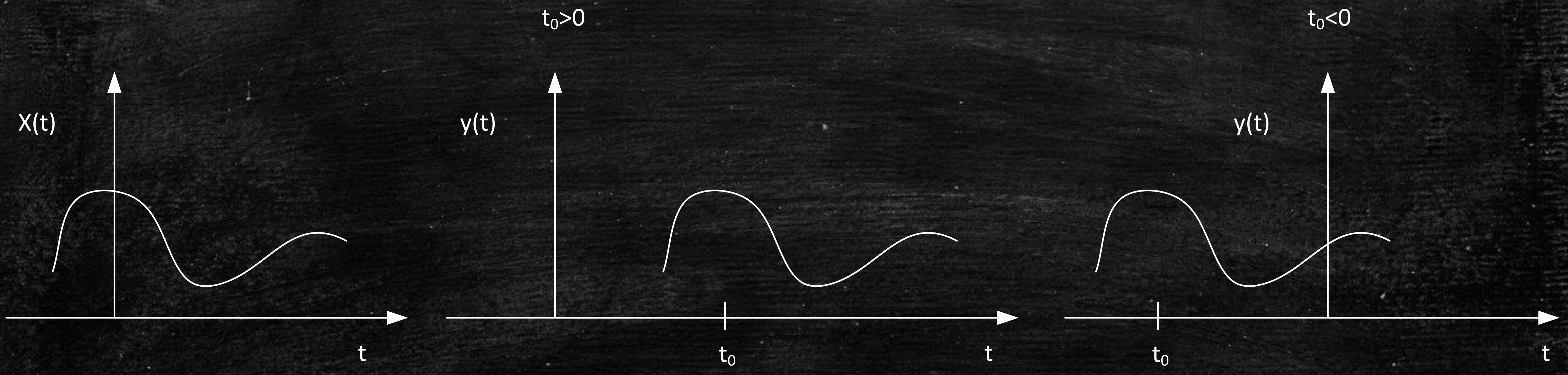
$$\lim_{t\to 0} sinc(t) = \lim_{t\to 0} \frac{\sin(\pi t)}{\pi t} = \frac{\pi \cos(\pi t)}{\pi}\bigg|^{t=0} = 1$$

  - Its zeros are in $\pm k\pi$

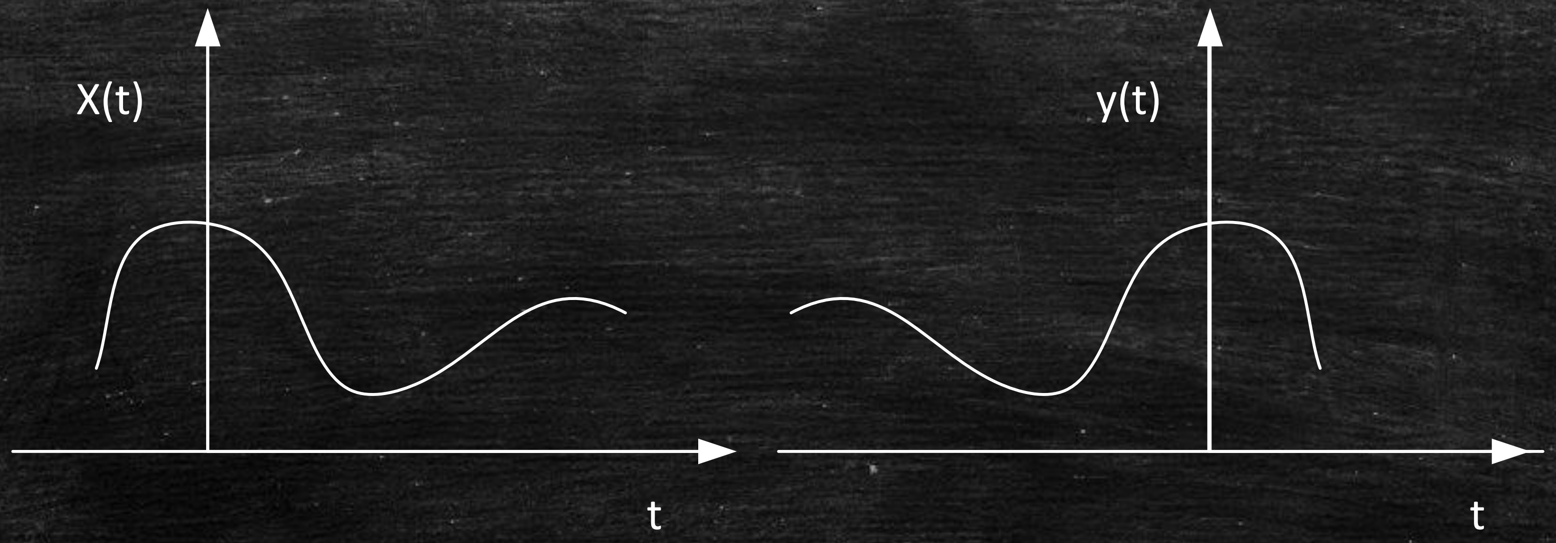


Convolution

- Signals can be added, subtracted, multiplied …
- Temporal shift: $y(t) = x(t - t_0)$



- Temporal inversion: $y(t) = x(-t)$



- Convolution: $z(t) = x(t) \otimes y(t) = \int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau$
  - Commutative property: $x(t) \otimes y(t) = y(t) \otimes x(t)$

$$y(t) \otimes x(t) = \int_{-\infty}^{\infty} y(\tau) x(t-\tau) d\tau \xrightarrow[\alpha = t-\tau]{} \int_{-\infty}^{\infty} y(t-\alpha) x(\alpha) d\alpha =$$
$$= x(t) \otimes y(t)$$

  - Associative property:

$$[x(t) \otimes y(t)] \otimes z(t) = x(t) \otimes [y(t) \otimes z(t)]$$

  - Distributive property:

$$x(t) \otimes [y(t) + z(t)] = [x(t) \otimes y(t)] + [x(t) \otimes z(t)]$$

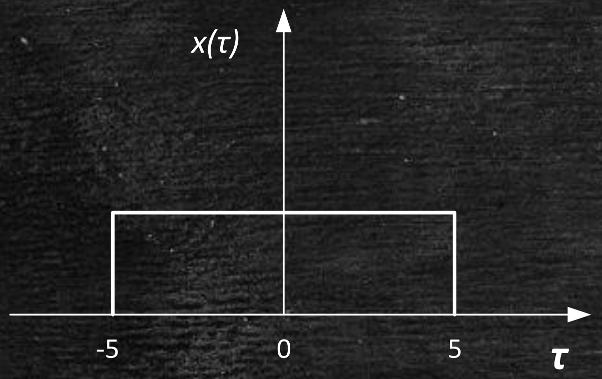
  - Differentiation:

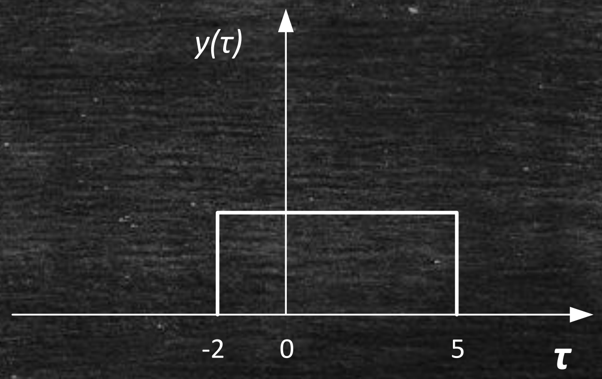
$$\frac{d}{dt}[x(t) \otimes y(t)] = \frac{dx(t)}{dt} \otimes y(t) = \frac{dy(t)}{dt} \otimes x(t)$$

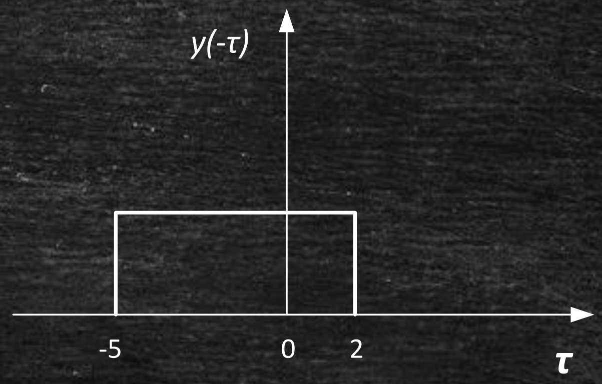
  - Area:

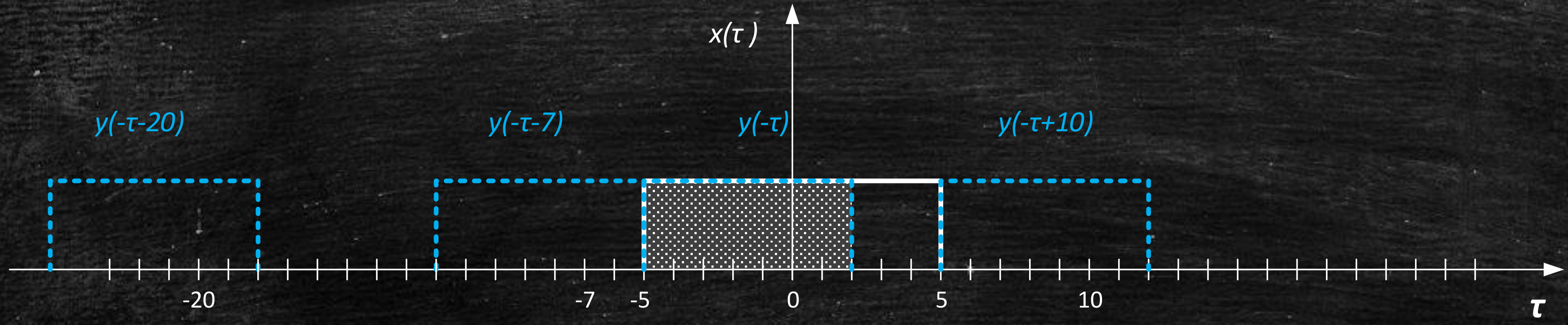
$$\int_{-\infty}^{\infty} x(t) \otimes y(t) dt = \int_{-\infty}^{\infty} x(t) dt \int_{-\infty}^{\infty} y(t) dt$$

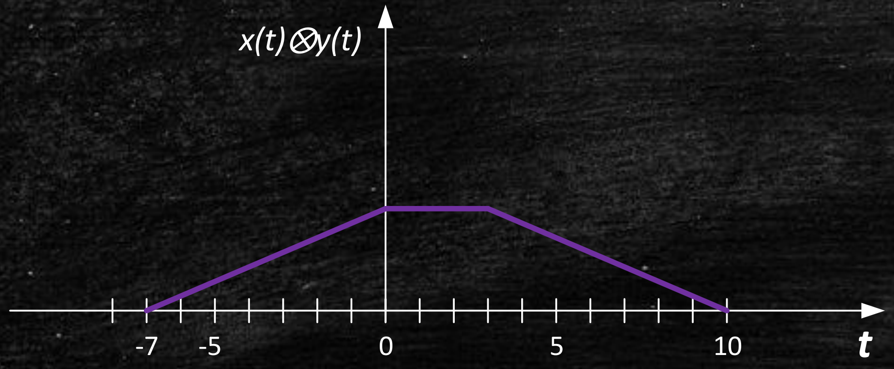
- Graphic example of convolution











$$x(t)\otimes y(t) = \begin{cases} 0, & t < -7 \\ A^2(t+7), & -7 \leq t < 0 \\ 7A^2, & 0 \leq t < 3 \\ -A^2(t-10), & 3 \leq t < 10 \\ 0, & 0 \leq t \end{cases}$$

- The convolution of a signal with a Dirac delta results in the same signal

$$x(t) * \delta(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau = x(t)$$

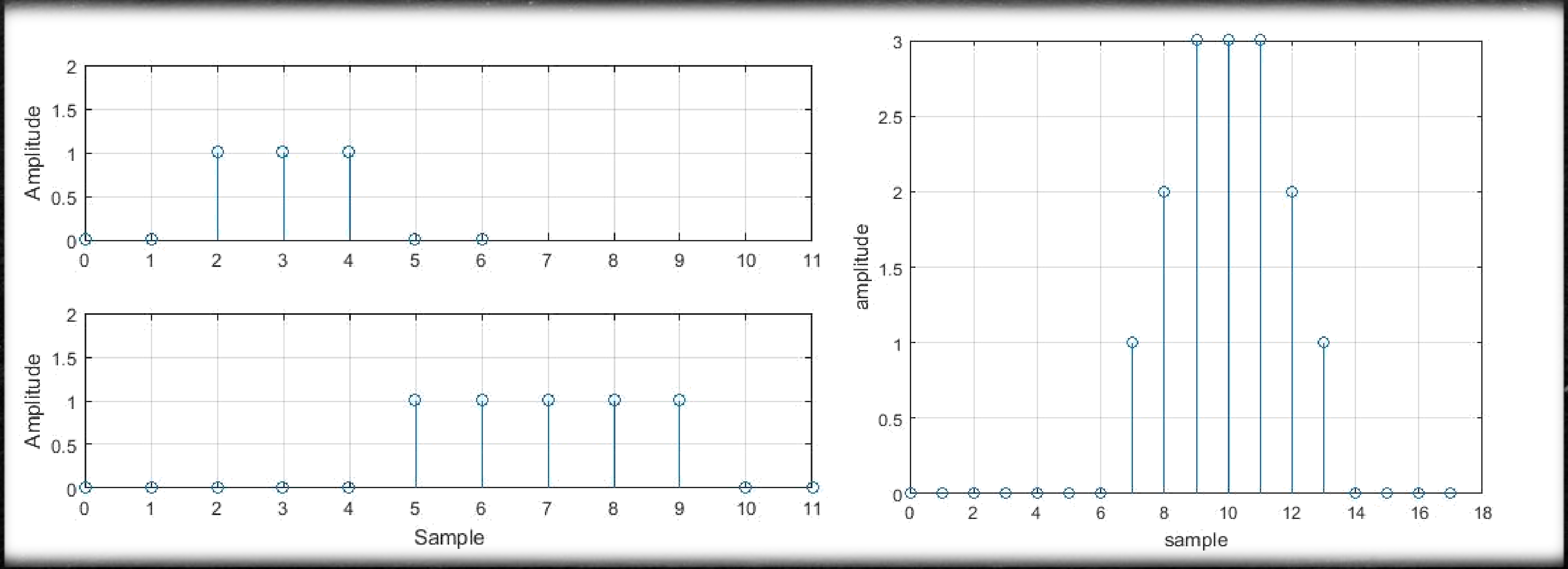
- If the Dirac delta has an offset in time the resulting convolution will have the same temporal offset

$$x(t) * \delta(t-t_0) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-t_0-\tau)d\tau = x(t-t_0)$$

- The discrete convolution between two sequences $x[n]$ and $y[n]$ can be described as

$$z[n] = x[n] \otimes y[n] = \sum_{n=-\infty}^{\infty} x[k]y[n-k]$$

- The length of the resulting sequence will be always the addition of the length of the convolved sequences minus 1



- Analytical way of proceeding
  $x[n] = [0\ 0\ 1\ 1\ 1\ 0\ 0\ ]\ ;\ y[n] = [0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ ]$

k	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	z[k]
y[k]							0	0	0	0	0	1	1	1	1	1	0	0							
x[-k]	0	0	1	1	1	0	0																		0
x[1-k]		0	0	1	1	1	0	0																	0
x[2-k]			0	0	1	1	1	0	0																0
x[3-k]				0	0	1	1	1	0	0															0
x[4-k]					0	0	1	1	1	0	0														0
x[5-k]						0	0	1	1	1	0	0													0
x[6-k]							0	0	1	1	1	0	0												0
x[7-k]								0	0	1	1	1	0	0											1
x[8-k]									0	0	1	1	1	0	0										2
x[9-k]										0	0	1	1	1	0	0									3
x[10-k]											0	0	1	1	1	0	0								3
x[11-k]												0	0	1	1	1	0	0							3
x[12-k]													0	0	1	1	1	0	0						2
x[13-k]														0	0	1	1	1	0	0					1

The Fourier Transform

- Mathematically the Fourier transform of a signal $x(t)$
- $X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, dt$
- $x(t)$ must verify the Dirichlet condition:
  - Have a finite number of maximum, minimum, and discontinuities in a finite interval
  - Must be an energy signal

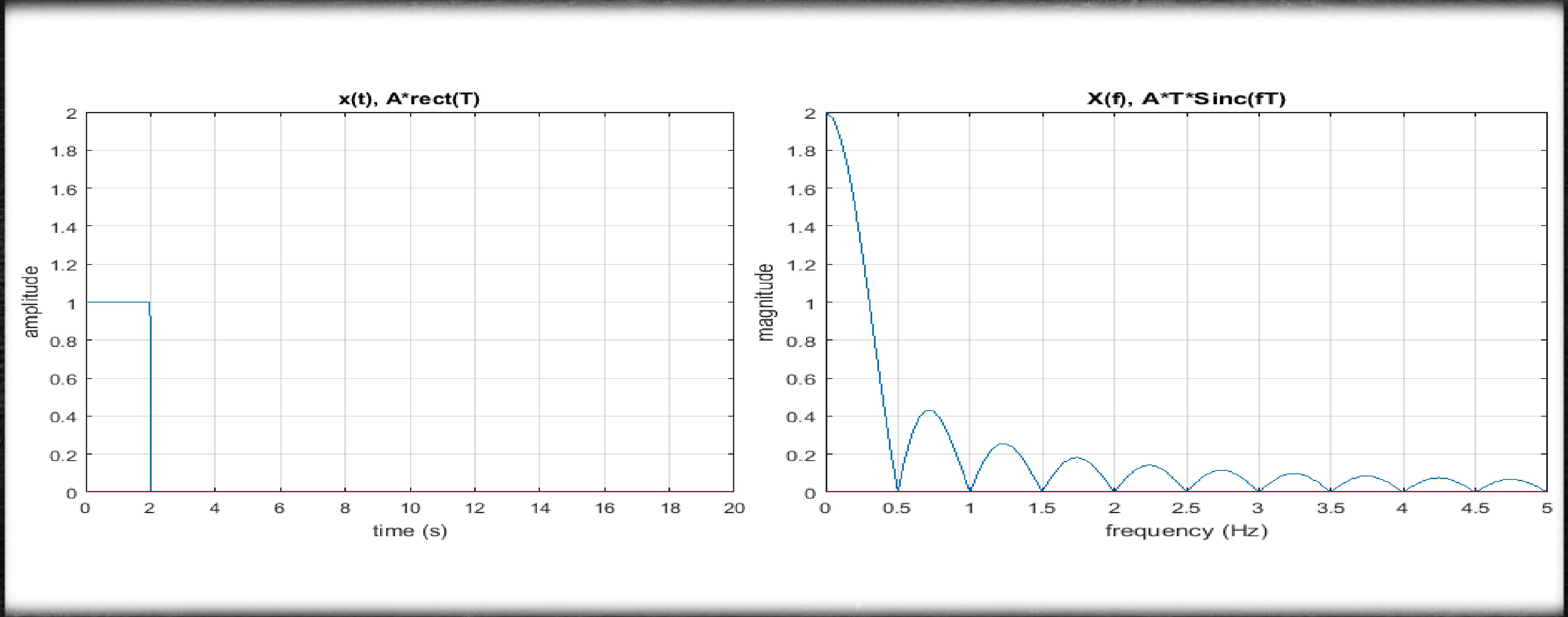
- The reverse Fourier transform has the following expression

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t}\, df$$

- Reverse Fourier transform allows to represent the signal in time as the weighted addition of complex exponentials
- Also if $x(t)$ is a real signal, $X(-f) = X^*(f)$

$$X(-f) = \int_{-\infty}^{\infty} x(t) e^{j2\pi f t}\, dt = \left(\int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, dt\right)^* = X^*(f)$$

- Fourier transform of a rectangular pulse



$$X(f) = \int_{-T_0/2}^{T_0/2} e^{-j2\pi f t}\, dt = \frac{T_0 \sin(\pi f T_0)}{\pi f T_0} = T_0 sinc(f T_0)$$

- Energy signals have their energy continuously distributed along the spectrum and not allocated in discrete frequencies

- The Fourier transform is linear, $X(f) = \mathcal{F}\{x(t)\}$ and $Y(f) = \mathcal{F}\{y(t)\}$: $\mathcal{F}\{ax(t) + by(t)\} = aX(f) + bY(f)$

$$\mathcal{F}\{ax(t) + by(t)\} = \int_{-\infty}^{\infty} [ax(t) + by(t)]e^{-j2\pi ft}\, dt$$

$$= a\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\, dt + b\int_{-\infty}^{\infty} y(t)e^{-j2\pi ft}\, dt = aX(f) + bY(f)$$

- Duality property

$$y(f) = \mathcal{F}\{x(t)\} = X(f) \Leftrightarrow x(-f) = \mathcal{F}\{y(t)\} = Y(f)$$

$$\mathcal{F}\{y(t)\} = \int_{-\infty}^{\infty} \mathcal{F}\{x(\tau)\}\ e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{\infty} X(t)\ e^{-j2\pi ft} dt \overset{F=-f}{\Longrightarrow} \int_{-\infty}^{\infty} X(t)\ e^{j2\pi Ft} dt = x(F) = x(-f)$$

- Time and frequency delay

$$x(t-t_0) \Leftrightarrow X(f)e^{j2\pi f t_0}$$
$$x(t)e^{j2\pi t f_0} \Leftrightarrow X(f-f_0)$$

- Convolution and product properties

$$x(t) \otimes y(t) \Leftrightarrow X(f)Y(f)$$
$$x(t)y(t) \Leftrightarrow X(f) \otimes Y(f)$$

$$\mathcal{F}\{x(t) \otimes y(t)\} = \int_{-\infty}^{\infty} x(t) \otimes y(t) e^{-j2\pi f t} dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau \, e^{-j2\pi f t} dt$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) y(t-\tau) \, e^{-j2\pi f t} d\tau dt \overset{\lambda = t-\tau}{\Longrightarrow}$$
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) y(\lambda) \, e^{-j2\pi f(\lambda+\tau)} d\tau d\lambda =$$
$$\int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \tau} d\tau \int_{-\infty}^{\infty} y(\lambda) \, e^{-j2\pi f \lambda} d\lambda = X(f)Y(f)$$

- Transform of the derivative of a function:

$$\frac{dx(t)}{dt} \Leftrightarrow j2\pi f X(f)$$

$$\frac{dx(t)}{dt} = \frac{d\mathcal{F}^{-1}\{X(f)\}}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\,df = \int_{-\infty}^{\infty} j2\pi f X(f)e^{j2\pi ft}\,df$$

- Transform of the conjugate of a function:

$$x^*(t) \Leftrightarrow X^*(-f)$$

$$\mathcal{F}\{x^*(t)\} = \int_{-\infty}^{\infty} x^*(t)e^{-j2\pi ft}\,dt = \left(\left(\int_{-\infty}^{\infty} x^*(t)e^{-j2\pi ft}\,dt\right)^*\right)^*$$

$$= \left(\int_{-\infty}^{\infty} x(t)e^{j2\pi ft}\,dt\right)^* \overset{F=-f}{\Longrightarrow} \left(\int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}\,dt\right)^* = X^*(F) = X^*(-f)$$

- Time scaling:

$$x(at) \Leftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-j2\pi f t}\, dt \overset{at=\tau}{\Longrightarrow} = \frac{1}{a}\int_{-\infty}^{\infty} x(\tau) e^{-j2\pi \frac{f}{a}\tau}\, d\tau = \frac{1}{a} X(\frac{f}{a})$$

  – This is only valid if $a > 0$, If a is a negative number also the limits change introducing an extra minus symbol and that's why the final result is divided by $|a|$

- We have already stated that periodical functions are very important in communications, but they don’t meet the Dirichlet conditions, they are power signals
- We use for them the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi}{T_0}kt}$$

- Existing a univocal relation between $x(t)$ and c_k

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j\frac{2\pi}{T_0}kt} dt$$

- Example : $x(t) = \sin(w_0 t) = \sin\left(\frac{2\pi}{T_0} t\right)$

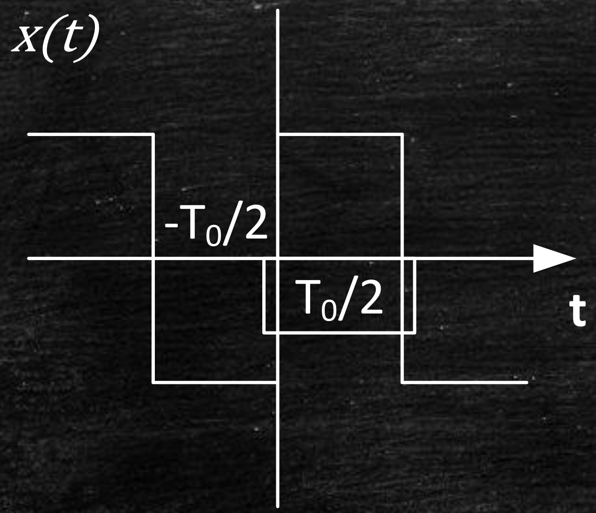
$$\sin\left(\frac{2\pi}{T_0} t\right) = -\frac{j}{2}\left[e^{j\frac{2\pi}{T_0}t} - e^{-j\frac{2\pi}{T_0}t}\right]$$

$$c_k = \begin{cases} -\frac{j}{2}, & k = 1 \\ \frac{j}{2}, & k = -1 \\ 0, & \forall k \neq \pm 1 \end{cases}$$

- Here we applied the definition of $x(t)$ being the addition of weighted complex exponentials and a trigonometry equality

- Example: periodic square signal

$$x(t) = \begin{cases} 1, -\frac{T_0}{2} \le t < 0 \\ -1, 0 \le t < \frac{T_0}{2} \end{cases} \quad c_k = \begin{cases} \frac{2}{j\pi k}, k = \pm 1, \pm 3, \pm 5, \ldots \\ 0, k = 0, \pm 2, \pm 4, \ldots \end{cases}$$



- Every periodic signal can be represented in frequency as its different c_k amplitudes at their corresponding frequency

- If $x(t)$ is real $c_k^* = c_{-k}$

$$c_k^* = \left(\frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\frac{2\pi}{T_0}kt}dt\right)^* = \frac{1}{T_0}\int_0^{T_0} x(t)e^{j\frac{2\pi}{T_0}kt}dt = c_{-k}$$

- If we gather the positive and negative values of k

$$c_k e^{j\frac{2\pi}{T_0}kt} + c_{-k}e^{-j\frac{2\pi}{T_0}kt} = c_k e^{j\frac{2\pi}{T_0}kt} + c_k^*\left(e^{j\frac{2\pi}{T_0}kt}\right)^*$$

$$= 2\Re e\left(c_k e^{j\frac{2\pi}{T_0}kt}\right) = 2\,|c_k|\cos\left(\frac{2\pi}{T_0}kt + \alpha_k\right)$$

- Where $|c_k|$ and α_k represent the module and phase of c_k

- The expression for $x(t)$ can be rewritten as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi}{T_0}kt} = c_0 + \sum_{k=1}^{\infty} c_k e^{j\frac{2\pi}{T_0}kt} + c_{-k} e^{-j\frac{2\pi}{T_0}kt} = c_0 + 2\sum_{k=1}^{\infty} |c_k| \cos\left(\frac{2\pi}{T_0}kt + \alpha_k\right)$$

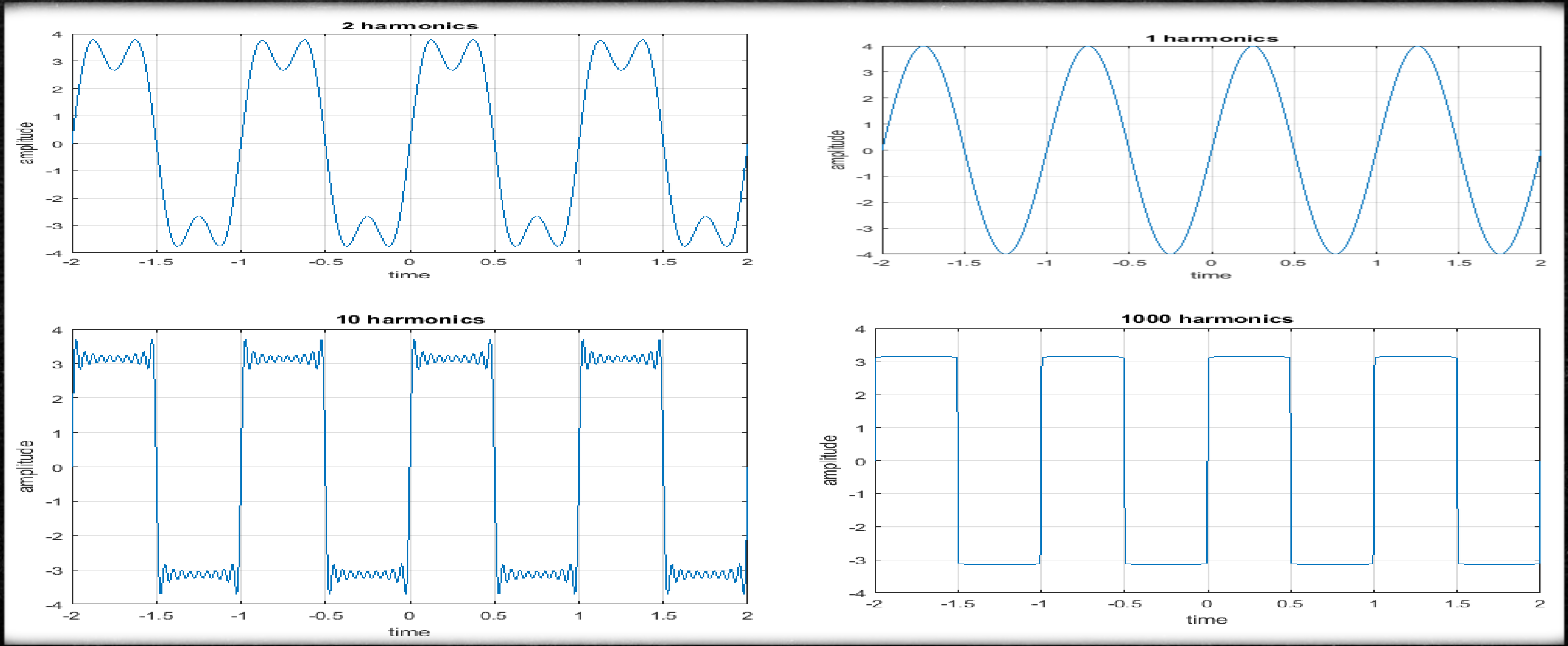
$$= c_0 + 2\sum_{k=1} a_k \cos\left(\frac{2\pi}{T_0}kt\right) + b_k \sin\left(\frac{2\pi}{T_0}kt\right)$$

- Where

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\cos\left(\frac{2\pi}{T_0}kt\right)dt \; b_k = \frac{1}{T_0}\int_0^{T_0} x(t)\sin\left(\frac{2\pi}{T_0}kt\right)dt$$

- c_0 represents the average value of $x(t)$, $f_1 = 1/T_0$ represents the fundamental frequency of the signal and the rest of them the different harmonics

- As an example, different number of harmonics of a square pulse



- For energy signals the energy can be calculated as:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

- For power signals:

$$P = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt = \sum_{-\infty}^{\infty} |c_k|^2$$

- In digital systems we apply what is called DFT: Discrete Fourier Transform
  - Do not confuse with DTFT (Discrete Time Fourier Transform) that is discrete in time ($x[n]$), but continuous in frequency

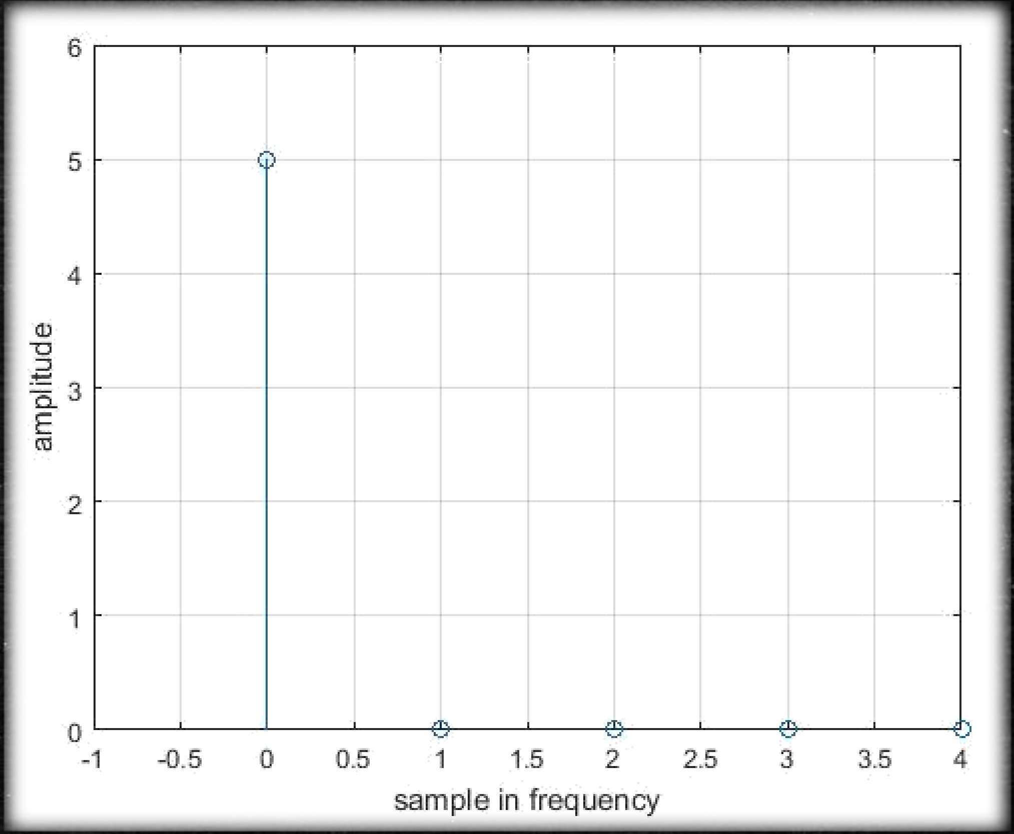
$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

  - Where N is the number of samples of the signal used

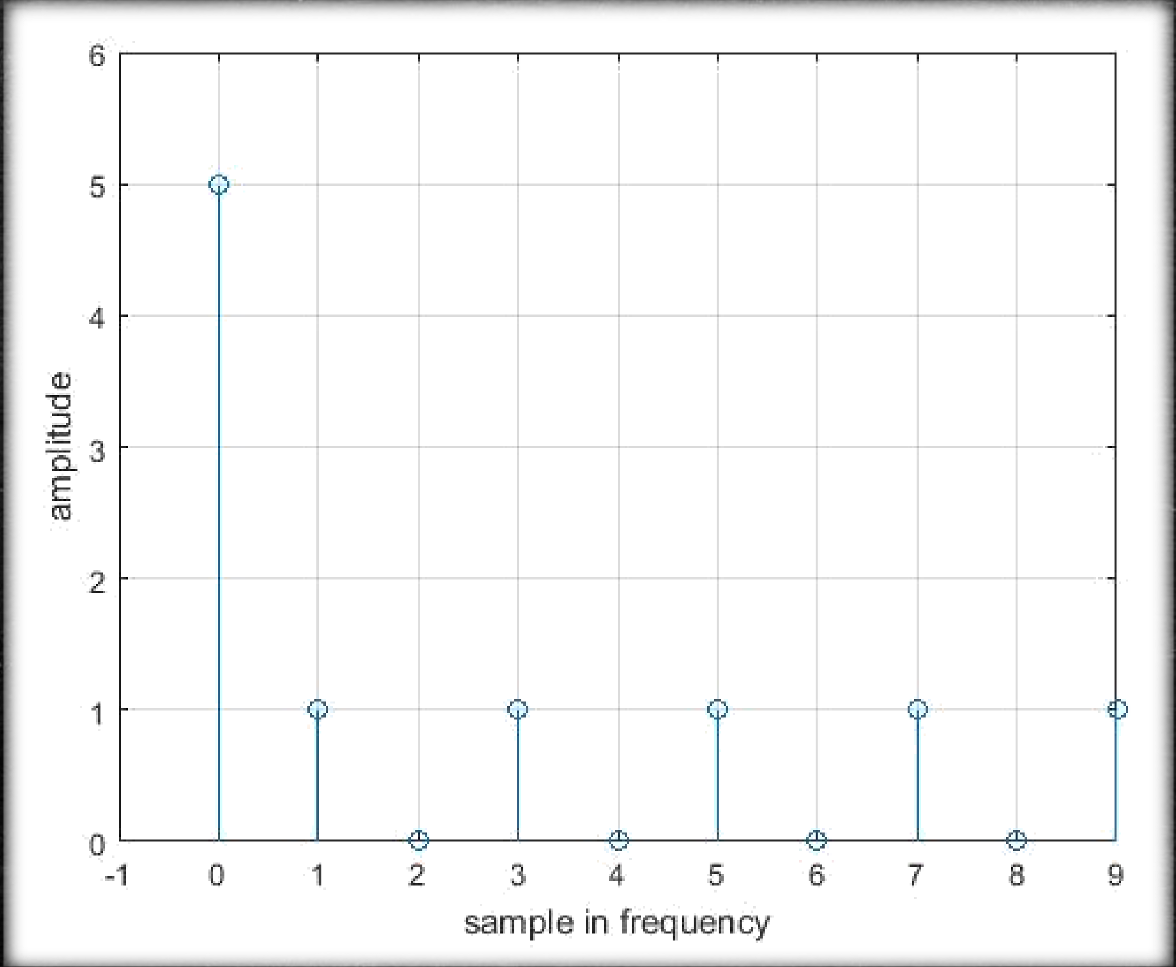
- The inverse Fourier transform has the following expression

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

- The DFT spectrum is periodic with period N
- The maximum frequency that can be represented is the sampling frequency, inverse of the sampling time , $F_s = 1/T_s$
- The frequency resolution will be F_s/N
- Example: $x[n] = [1\ 1\ 1\ 1\ 1]$

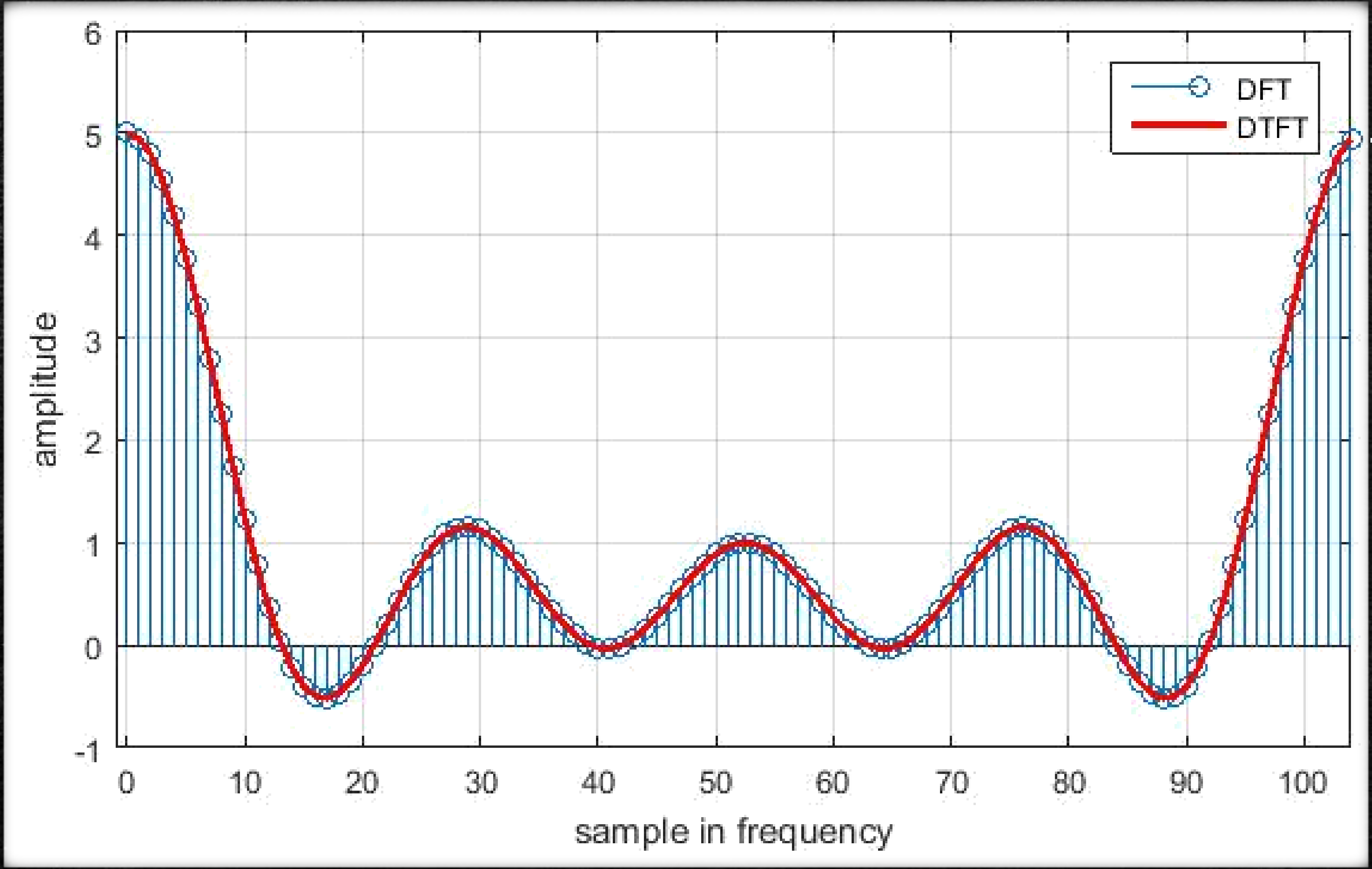


- But what if we change a bit the example? $x[n] = [1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0]$



- This technique is known as zero padding and changes the resolution in the frequency axis but not the frequency resolution of the spectral components that are dependent only of the M non zero samples

- If we further increase the zero padding to 100 samples



Sampling and quantization

- In nature as we perceive it, physical magnitudes that can be measured are
  - Continuous in time
  - Continuous in amplitude
- We live in the analogue "world"
- But to work with computers, microprocessors, etc. we cannot have a infinite accuracy, we work with bits (representing float, integer, ...)
- A variable is digital when only can take certain values of a finite group, $x_D \in X_D \; [x_0, x_1, x_2, \dots]$

THANKS !

Any questions?

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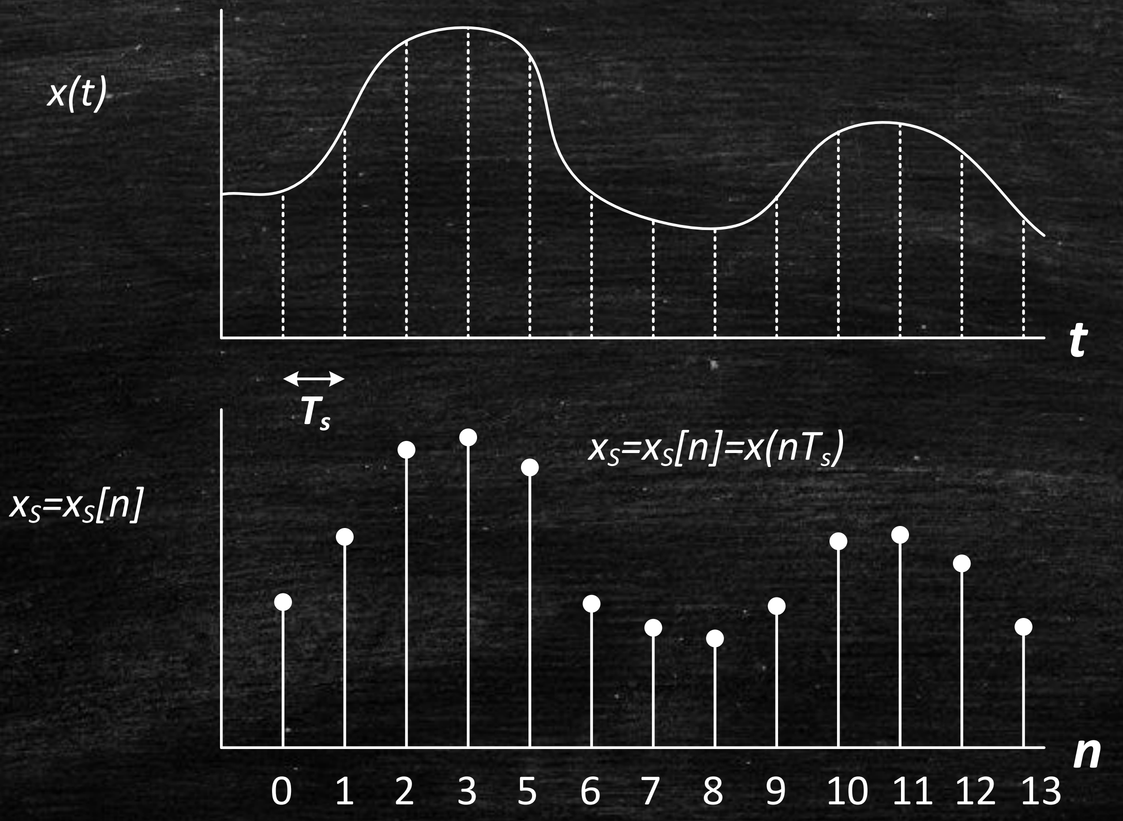
- As an example, a natural binary variable of 8 bits $x_D \in X_D$ $[0,1,2, \ldots, 255]$
- The possible values can be ordered following a sequence so each value is represented by the place it occupies, $x_D = x_D[n]$
- If the n index represents an ordered temporal occurrence, then x_D is a digital signal
- It is possible to represent a signal $x(t)$ by means of a sequence of numbers $x_D = x_D[n]$

- The resulting digital signal
  - Is discrete and each index n represents a time instant
  - Is discrete, takes values from a finite set
  - Can be digitally stored
- In order to digitalize a signal two operations must be performed: sampling and quantization
- A digital signal $x_D = x_D[n]$, under certain conditions can be transformed again into the original $x(t)$
  - To get the digital version, x_D, ADC are used
  - To transform a digital signal into analogue, DAC are used

- Sampling a signal is to register its value every certain period of time
- Usually the time between samples, sampling time (T_s), is constant and defines also the sampling frequency ($f_s = 1/T_s$)



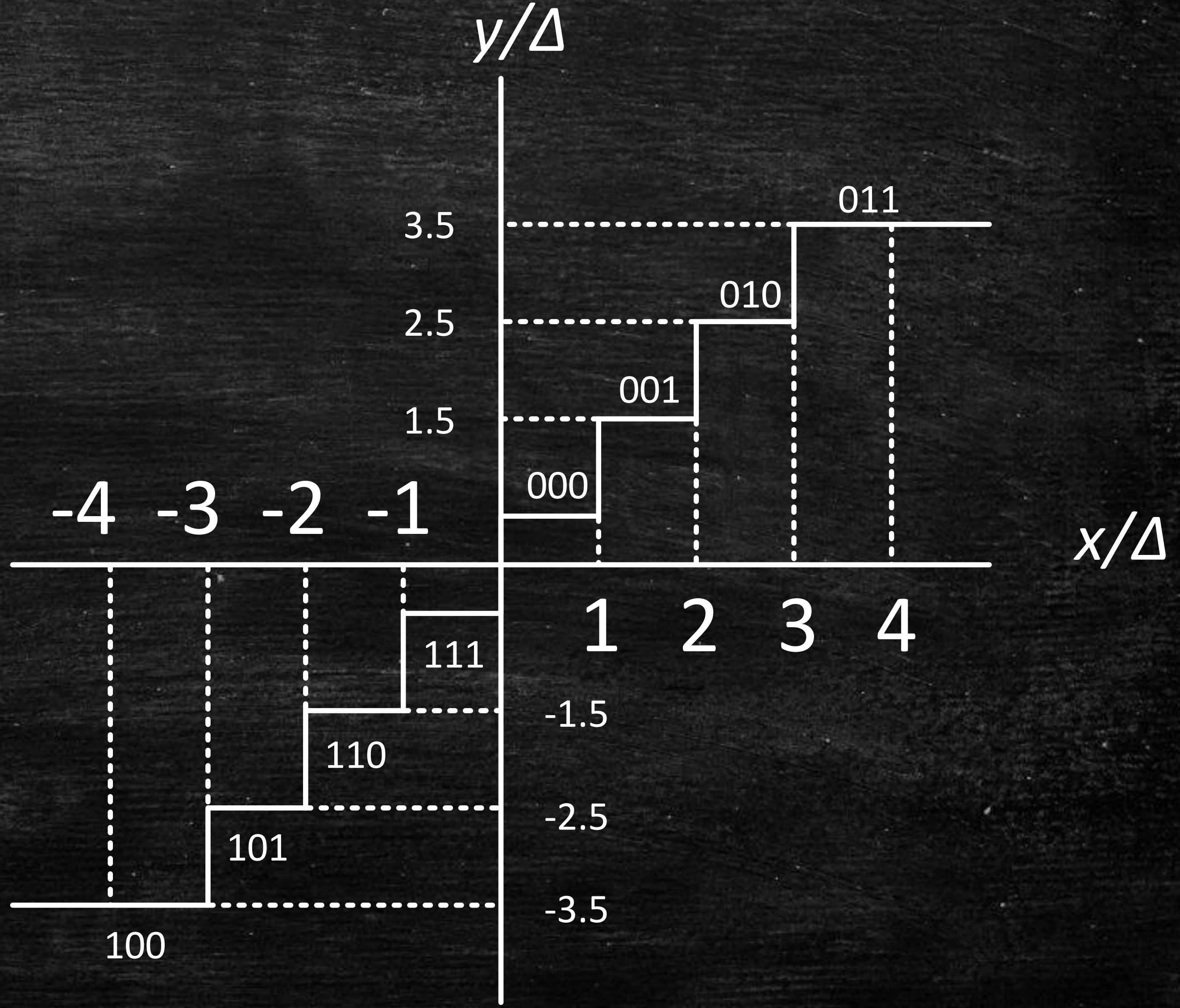
- As said previously the digital signals have a finite number of possible values, $x_D \in X_D$, usually the sampled values will not correspond with one of these possible values
- It will be necessary to assign one of them to the sample following some strategy
- In general it can be said that $x_D = Q\big(x(nT_s)\big) = Q(x_S[n])$, and $Q(\cdot)$ can take different forms
  - Round
  - Truncate
  - …

- Linear quantifiers have a stair shaped output form like the following

- Δ represents the quantization step
- In the example 3 bits quantization is used
- The example uses the following quantization rule:

$$Q(x) = \Delta\left(\left\lfloor\frac{x}{\Delta}\right\rfloor + \frac{1}{2}\right)$$

- 0 cannot be represented



- By quantifying the signal we are introducing an error

- The error we introduce is the difference between output and input, $e = y - x = Q(x) - x$

- Inside the quantization interval the error is bounded in the interval $[-\Delta/2, \Delta/2]$

- In general $\Delta = \frac{x_{max} - x_{min}}{2^m}$, where m represents the number of bits used in the quantization